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It is shown that if one can perform a restricted set of fast manipulations on a quantum system, one can implement a large class of dynamical evolutions by effectively removing or introducing selected Hamiltonians. The procedure can be used to achieve universal noise-tolerant control based on purely unitary open-loop transformations of the dynamics. As a result, it is in principle possible to perform noise-protected universal quantum computation using no extra space resources.

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The desire to shape quantum evolution according to precisely controlled dynamics is shared by various areas of contemporary physics and engineering [1]. A problem commonly encountered in the task of controlling the dynamical behavior of a quantum system is the need for removing unwanted interactions present in the full Hamiltonian: historically, one of the first solutions was provided by Nuclear Magnetic Resonance spectroscopy, where a variety of *decoupling* techniques have been developed to simplify spectra by effectively eliminating selected contributions to the nuclear Hamiltonian [2]. Once suppression of a given term is obtained, the corresponding Hamiltonian may no longer be directly available for control. From the perspective of attaining universal dynamical control, this raises the question of devising ways for introducing or re-introducing Hamiltonian control compatible with the prescribed decoupling action.

A striking example is the case of an open quantum system, where the coupling to the environment is responsible for inducing quantum decoherence and dissipation processes, thereby corrupting the original unitary dynamics [3]. The demand for universal control strategies of noise-decoupled quantum systems, able to effectively reject environmental noise while still ensuring full control capabilities, has been heightened tremendously due to the challenge of implementing quantum computation [4]. In spite of the beautiful and powerful advancements made in the theory of quantum error correction [5], fault-tolerant error correction [6] and concatenated coding [7], practical exploitation of these results is still seriously constrained by the amount of extra space resources required [8].

In this Letter, we address the general problem of *open-loop* controllability of a decoupled quantum system: the controller is assumed to apply time-dependent potentials without ever measuring the actual state of the system. We introduce *programming* procedures for combining the desired control action with the decoupling operations and identify the conditions under which universal control over the effective decoupled dynamics is retained. As a consequence, we demonstrate the possibility of achieving noise-tolerant control of open quantum systems solely based on unitary manipulations which do not require ancillary resources.

Decoupling.– We begin by recalling the essential ingredients of decoupling in the language of [9]. Let H be the Hamiltonian of a quantum system living in a Hilbert space $\mathcal{H}$. Suppose we have the capability of performing instantaneously a certain set of unitaries, meaning that the corresponding set of Hamiltonians can be turned on for negligible amounts of time τ with (ideally) arbitrarily large strength. We shall term such impulsive full-power control operations as *bang-bang* (b.b.) controls [9,10]. If U is a rotation that can be implemented b.b., it is conceivable that $U^{-1} = U^\dagger$ can be realized b.b. as well. In this case, the set of realizable b.b. operations is a subgroup $\mathcal{G}_{b.b.}$ of the full group $\mathcal{U}(\mathcal{H})$ of unitary transformations over $\mathcal{H}$. A *decoupler* on $\mathcal{H}$ operates by iterating the system through a cyclic time evolution which judiciously combines sequences of b.b. operations with free evolutions under the natural Hamiltonian H. Accordingly, a decoupler shall be characterized by a finite group $\mathcal{G} \subseteq \mathcal{G}_{b.b}$ of b.b. operations (*decoupling group*) together with a known time scale T_c (*cycle time*) determining the duration of a single cycle. Full knowledge of the decoupler operations, including the exact sequence of group operations g_j, $j = 0, \ldots, |\mathcal{G}|-1$, $|\mathcal{G}| \equiv \mathrm{ord}(\mathcal{G})$, and their temporal separation Δt, may not be available from the beginning (black-box decoupler).

What does a decoupled evolution look like ? A convenient picture is provided by average Hamiltonian theory [2,9]. Consider a slicing of a given evolution time T as $T = NT_c \equiv N|\mathcal{G}|\Delta t$, and let $U_0(\Delta t) = \exp(-iH\Delta t)$ be the free propagator. The evolution in the presence of the decoupler is uniquely determined by the sequence of group transformations $\{g_j\}$ over a single cycle time:

$$U(T_c) = \prod_{j=0}^{|\mathcal{G}|-1} g_j^\dagger U_0(\Delta t) g_j \equiv e^{-iH_{eff}T_c} \; , \tag{1}$$

H_{eff} denoting the resulting effective Hamiltonian. In the ideal limit of arbitrarily fast cycle time $T_c \to 0$, with $N \to \infty$ in such a way that $NT_c = T$, H_{eff} approaches

$$H \mapsto H_{eff} = \frac{1}{|\mathcal{G}|} \sum_{g_j \in \mathcal{G}} g_j^\dagger \, H \, g_j \equiv \Pi_{\mathcal{G}}(H) \; . \tag{2}$$

Eq. (2) defines a quantum operation on the space $\mathrm{End}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. Physically, H_{eff} represents the leading contribution to the average Hamiltonian describing the motion of the system under the influence of the decoupling field, the j-th term in the sum (2) being identical to the so-called *toggling frame* Hamiltonian during the j-th cycle subinterval [2]. From a geometric point of view, the map $\Pi_{\mathcal{G}}$ introduced in (2) can be identified with the projector on the so-called *centralizer* $Z(\mathcal{G})$ in $\mathrm{End}(\mathcal{H})$ [9,11]:

$$Z(\mathcal{G}) = \{\, \mathcal{O} \,|\, [\mathcal{O}, g_j] = 0 \;\; \forall g_j \in \mathcal{G} \,\} = \Pi_{\mathcal{G}}(\mathrm{End}(\mathcal{H})) \,. \quad (3)$$

$Z(\mathcal{G})$ is a subalgebra of $\mathrm{End}(\mathcal{H})$. We shall denote by $Z_H(\mathcal{G})$ the subspace of Hermitian operators belonging to $Z(\mathcal{G})$. Eq. (3) allows a direct interpretation of the decoupler action in terms of symmetry properties: since $H_{eff} = \Pi_{\mathcal{G}}(H)$, the decoupled evolution is *symmetrized* according to the group $\mathcal{G}$ [11]. All the components of the dynamics generated by H, which are *not* invariant under the group $\mathcal{G}$, are effectively averaged out.

The group-theoretical prescription (1) translates directly to pulse control, a control cycle involving a sequence of decoupling pulses $D_j = g_j g_{j-1}^{\dagger}$, $j = 1, \ldots, |\mathcal{G}|$, separated by delays of free evolution Δt and fulfilling $D_{|\mathcal{G}|} \ldots D_2 D_1 = \mathbb{1}$ by cyclicity. The limit $T_c \rightarrow 0$ cannot be met exactly in practice. While symmetry properties suffice to specify the output of a given decoupler, time scales are crucial in determining how good the decoupler performs in a realistic scenario where both the pulse duration τ and the cycle time T_c are finite. Qualitatively speaking, the projection on the centralizer (2) will tend to *symmetrize interactions whose typical correlation times are long on the time scale determined by* T_c. If τ_c represents the shortest correlation time associated with the unwanted interactions, decoupling will be effective under the hierarchy of time scales $\tau \ll \Delta t \leq T_c \ll \tau_c$ [9]. Effects due to finite pulse duration are expected to be of order $\mathrm{O}(\tau/\tau_c)$, while cycle time corrections scale with a ratio $\mathrm{O}(T_c/\tau_c)$ [9]. It is worth stressing that T_c determines the minimum time scale over which decoupling is guaranteed. Thus, the original continuous-time dynamics under H is replaced by a *stroboscopic* time development under H_{eff} with a natural time unit equal to T_c.

For open quantum systems, the above description comprises two coupled subsystems S and B with associated Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$, the S-component representing the system of physical interest (*e.g.*, the computational degrees of freedom of a quantum computer). If $H = H_S + H_B + H_{SB}$ denotes the total Hamiltonian, environmental noise is introduced through a set of (traceless) error operators E_α in the interaction Hamiltonian, $H_{SB} = \sum_\alpha E_\alpha \otimes B_\alpha$, B_α denoting bath operators. We require the linear space $\mathcal{E} = \mathrm{span}\{E_\alpha\}$ to be finite dimensional. An error space $\mathcal{E}$ will be *correctable* by any decoupler ensuring that $\Pi_{\mathcal{G}}(\mathcal{E}) = 0$. The appropriate correlation time τ_c is related to the memory time of the reservoir B [9]. Note that this necessarily implies a non-Markovian error scenario.

The capability of successfully decoupling the system S from the effects of the environment is only a first step to achieving noise-tolerant control of S-degrees of freedom. Can we do better?

Programming.– The presence of a decoupler imposes restrictions on the manipulations available to further control the system dynamics. Those restrictions are in the form of both *symmetry* and *timing* constraints. Suppose, for instance, that a Hamiltonian A is applied over a time interval long compared to T_c. Then the evolution generated by A may be quenched completely if $A \notin Z_H(\mathcal{G})$. If, instead, the action corresponding to A is implemented in the form of b.b. rotations, it is easy to check that, unless the insertion points are chosen carefully, any extra pulse interferes with the decoupler operations, thereby spoiling noise averaging. This can be avoided, for instance, by making sure that the b.b. operations are inserted *in between* cycles only. In general, indiscriminate application of control operations will not lead to the expected result. The goal of programming is to characterize the degree of control attainable compatibly with the desired decoupling action. This is influenced by three factors: available knowledge of the error space $\mathcal{E}$; available knowledge of the decoupler operations; available control resources. In particular, since switching on/off strong interactions for short times is difficult in practice, a relevant criterion will be trying to keep the number of required b.b. operations to a minimum.

Let us first consider the case where the error space $\mathcal{E}$ is known and a decoupling group $\mathcal{G}$ exists with a non-trivial centralizer, $Z(\mathcal{G}) \neq \{\mathbb{1}\}$, a situation corresponding to selective averaging according to [9]. By virtue of (2), it is always possible to apply slowly any Hamiltonian $A \in Z_H(\mathcal{G})$ in parallel with the decoupler. However, since the unitary evolutions generated by such Hamiltonians also lie by construction in the centralizer of $\mathcal{G}$, universality on the full Hilbert space cannot be achieved by purely exploiting this kind of *weak (strength)/ slow (switching) control*. Supplementary coding methods are demanded, which shall be discussed elsewhere [12]. Instead, we focus here on the idea of combining slow control from the set $Z_H(\mathcal{G})$ with suitable fast manipulations available in addition to the decoupling ones.

A relatively straightforward situation occurs when the group $\mathcal{G}_{b.b.}$ of realizable b.b. operations is large enough to accommodate *two* distinct decouplers with known (possibly different) cycle times, *i.e.*, $\Pi_{\mathcal{G}}(\mathcal{E}) = 0$, $\Pi_{\tilde{\mathcal{G}}}(\mathcal{E}) = 0$, with $\mathcal{G}, \tilde{\mathcal{G}} \subset \mathcal{G}_{b.b.}$. Notice that if $\mathcal{G}$ and P are respectively a decoupling group and a unitary transformation, $P \notin \mathcal{G}$, $P \notin Z(\mathcal{G})$, then $\tilde{\mathcal{G}} = P^{\dagger}\mathcal{G}P$ implements a *twisted decoupler* of the same order provided $P^{\dagger}\mathcal{E}P = \mathcal{E}$. Suppose now we can apply Hamiltonian $A \in Z_H(\mathcal{G})$ for a time interval $\Delta T_1 = N_1 T_{c1}$ (in parallel with decoupler 1), Hamiltonian $B \in Z_H(\tilde{\mathcal{G}})$ for a time interval $\Delta T_2 = N_2 T_{c2}$ (in parallel with decoupler 2), etc., (*e.g.*, one could have $A = \Pi_{\mathcal{G}}(H)$, $B = \Pi_{\tilde{\mathcal{G}}}(H)$). Then, by using standard universality results [13], any $U = e^L$ could be created, where L belongs to the Lie algebra generated by iA, iB under commutation. Accordingly, universal control over the decoupled dynamics is achieved whenever this algebra amounts to the whole Lie algebra of anti-Hermitian operators.

Even if only a single decoupler $\mathcal{G}$ is available, a similar strategy can be mimicked through a simple trick. Suppose that, in addition to decoupling pulses in $\mathcal{G}$, we can perform on the system a b.b. rotation $P \in \mathcal{G}_{b.b.}$. What can we do with this capability ? Assuming that we are able to *synchronize* operations with the cycle time, we can make the system effectively evolve according to a *transformed* average Hamiltonian. Let $\Delta T = NT_c$ be a given time window, with decoupled evolution ruled by the Hamiltonian (2). Imagine now inserting a pulse P immediately before the beginning of ΔT, followed by a pulse $P^\dagger$ synchronized with the end of ΔT. Then evolution over ΔT can be described in terms of a new average Hamiltonian $\tilde{H}_{eff} = P^\dagger \Pi_{\mathcal{G}}(H) P$ *i.e.*,

$$\tilde{H}_{eff} = \frac{1}{|\tilde{\mathcal{G}}|} \sum_{\tilde{g}_j \in \tilde{\mathcal{G}}} \tilde{g}_j^\dagger \, \tilde{H} \, \tilde{g}_j \equiv \Pi_{\tilde{\mathcal{G}}}(\tilde{H}) \;, \qquad (4)$$

where $\tilde{H} = P^\dagger H P$ and a twisted decoupling group $\tilde{\mathcal{G}} = P^\dagger \mathcal{G} P$ has been defined, with associated centralizer $Z(\tilde{\mathcal{G}}) = P^\dagger Z(\mathcal{G}) P$. Thus, the net effect of the two programming pulses $P, P^\dagger$ amounts to implement decoupling according to $\tilde{\mathcal{G}}$, noise averaging being retained since *both* the original decoupling group $\mathcal{G}$ and the error space $\mathcal{E}$ are simultaneously rotated. Clearly, one has to ensure that $P \notin Z(\mathcal{G})$ in order to steer the effective Hamiltonian out of the original centralizer $Z(\mathcal{G})$.

Let A now be, as above, a realizable Hamiltonian in the centralizer of $\mathcal{G}$ and let $B = P^\dagger A P$ denote its rotated counterpart. Then, by alternating evolution periods according to A and B, the latter being obtained by inserting pairs of pulses $P, P^\dagger$ with appropriate timing, it is possible in principle to obtain any Hamiltonian in the algebra generated by A, B under commutation. The reasoning is easily extended to the case where a given choice of interactions is realizable in $Z_H(\mathcal{G})$. In practice, the advantageous feature of this scheme is that by performing a single extra b.b. operation, a new repertoire of Hamiltonians becomes effectively available for slow control in the centralizer $Z(\tilde{\mathcal{G}})$. In the generic case, under the conditions given in [13], any desired unitary transformation will be reachable in principle, thereby implying complete control of the decoupled evolution.

Note that knowledge of the exact decoupling sequence has not been exploited so far, implying validity of the previous schemes even for black-box decouplers. If detailed information on the decoupler operations is available, this knowledge can be used to devise alternate control schemes implying less stringent resources. Suppose that we want to reintroduce control by some Hamiltonian $B \notin Z_H(\mathcal{G})$. If $e^{iB} \in \mathcal{G}_{b.b.}$, one could always, in principle, exploit the freedom of inserting such a b.b. pulse at the beginning and/or the end of decoupling cycles without affecting decoupling itself. Actually, it is possible to replace the the b.b. requirement with a weaker assumption, by imagining that the strength of B cannot be made arbitrarily large but B can still be turned on and off arbitrarily fast. In other words, let us assume a form of *weak (strength)/ fast (switching) control* whereby Hamiltonians can be modulated at the same rate as the b.b. control within a cycle. Then to reintroduce control according to B over a time interval $\Delta T = NT_c$, it suffices to turn it on during the $\mathbb{1}$-frame subinterval of each decoupling cycle. The evolution is ruled by the effective Hamiltonian

$$\tilde{H}_{eff} = \frac{1}{|\mathcal{G}|} \Big(\sum_{g_j \in \mathcal{G}} g_j^\dagger \, H \, g_j + B \Big) = \Pi_{\mathcal{G}}(H) + \frac{1}{|\mathcal{G}|} B \;, \qquad (5)$$

which acquires a component along $B \notin Z_H(\mathcal{G})$. Strength reduction for such a Hamiltonian can be avoided if an enlarged set of interactions is amenable of fast switching: one just turns on a Hamiltonian $B_j = g_j B g_j^\dagger$ during the j-th subinterval of each cycle, the overall effect being elimination of the $|\mathcal{G}|^{-1}$-factor in front of B.

Using the above methods, controlled evolutions can be designed by both letting the system evolve under the action of the decoupler alone and by incorporating modified decoupling cycles to displace the effective Hamiltonian out of $Z(\mathcal{G})$ as in (5). The issue of complete control can now be addressed by looking at the combined repertoire of interactions available for slow control in the centralizer, *e.g.*, $A = \Pi_{\mathcal{G}}(H) \in Z_H(\mathcal{G})$, together with the ones capable of supporting fast modulation, *e.g.*, Hamiltonian B considered above. Again, the conditions established in [13] provide a necessary and sufficient criterion for universality.

Let us briefly comment on the situation where no knowledge is available on the error space $\mathcal{E}$. In this case, decoupling can be achieved only by maximal averaging [9], so that the effective Hamiltonian is a trivial c-number, $H_{eff} = \lambda \, \mathbb{1}$. Since for decoupling groups with this property $Z(\mathcal{G}) = \{\mathbb{1}\}$, control schemes based on multiple or twisted decouplers are not useful anymore. In principle, one could still attain complete control in two circumstances: either $\mathcal{G}_{b.b.}$ contains a universal set of operations, which have to be performed synchronously with the decoupler clock T_c; or a universal set of Hamiltonians can be switched fast. Even with this option, the minimum number of required b.b. operations, $|\mathcal{G}| = (\dim(\mathcal{H}_S))^2$ [9], may be very large for relevant systems, strength losses in effective Hamiltonians (5) becoming possibly quite significant. We turn now to analyze more specifically the case of quantum computation.

Universal Quantum Computation.– Consider a quantum computer made of K qubits, $\mathcal{H}_S \simeq (\mathbb{C}^2)^{\otimes K}$, and assume that the relevant coupling to the environment can be accounted by a *linear* interaction of the form

$$H_{SB} = \sum_{a,i} \sigma_a^{(i)} \otimes B_a^{(i)} \;, \quad a = x, y, z; \; i = 1, \ldots K \;, \qquad (6)$$

for suitable bath operators $B_a^{(i)}$. The above coupling encompasses various models of interest, with error space spanned by combinations of single-qubit operators. Following the classification of [14], $\dim(\mathcal{E}) = 3K$ for independent decoherence where $\{E_\alpha\} = \{\sigma_a^{(i)}\}$, while $\dim(\mathcal{E}) = 3$ in the opposite limit of collective decoherence with global generators $\{E_\alpha\} = \{\sum_i \sigma_a^{(i)}\}$, intermediate situations occurring with cluster decoherence. Selective decoupling of the quantum register from H_{SB} requires a minimum of 3 b.b. operations *i.e.,* $|\mathcal{G}| = 4$. A convenient choice is $\mathcal{G} = \{\mathbb{1}, \otimes_{i=1}^K \sigma_a^{(i)}\}$, in which case the g_j's correspond to collective π-rotations and $Z(\mathcal{G})$ is generated by bilinear interactions of the form $\sigma_a^{(i)}\sigma_a^{(j)}$ [9,15].

We sketch now the application of the programming schemes described above. Suppose that, in addition to the $\mathcal{G}$-decoupler, we are equipped with a second decoupler based on a group $\tilde{\mathcal{G}}$, where for instance $\sigma_x^{(i)}$ is interchanged with $\sigma_z^{(i)}$ and $\sigma_y^{(j)}$ with $\sigma_z^{(j)}$ (say, $i = 1, j = 2$). Obviously, $\Pi_{\tilde{\mathcal{G}}}(\mathcal{E}) = 0$. Then almost any set of gates $U_A = e^{iAt_A}$, $U_B = e^{iBt_B}$ will be universal over $\mathcal{H}_S$, noise-suppression being preserved if Hamiltonians $A \in Z_H(\mathcal{G})$, $B \in Z_H(\tilde{\mathcal{G}})$ are applied in parallel to the decouplers $\mathcal{G}$, $\tilde{\mathcal{G}}$ respectively. Since the latter groups are connected by a double $\pi/2$-pulse $P = \exp(-i(\pi/4)\sigma_y^{(1)})\exp(-i(\pi/4)\sigma_x^{(2)})$, the same result can be reached if the realizable group of b.b. operations only includes $\mathcal{G}_{b.b.} = \{\mathcal{G}, P, P^\dagger\}$, but we have the capability of inserting $P, P^\dagger$ pulses synchronized with T_c. Finally, in the case that full knowledge is available of the decoupling sequence, a constructive result can be given. Note that the centralizer $Z_H(\mathcal{G})$ contains the bilinear Heisenberg couplings $\sum_a \sigma_a^{(i)}\sigma_a^{(j)}$ enabling one to to implement swapping between any pair of qubits [16]. Since controlled-NOT gates can be assembled as a sequence of "square-root swaps" and single-qubit operations [16], universal quantum logic can be performed if we have access to fast modulation of single-qubit Hamiltonians in addition to the required two-qubit interactions in the centralizer. A similar conclusion was conjectured in [17].

Whether the proposed approach can be viable in realistic situations will strongly depend on the details of the system and the environmental noise, as well as on the sophistication of the available technology. Bang-bang control is potentially suitable for NMR quantum computation, as long as b.b. pulses are able to be effected with a bandwidth small on the scale of the required spectral resolutions. At present, an experimental demonstration of b.b. control has been reported for all-optical quantum circuits [18]. Despite the challenges involved, the appeal of limited space resources may stimulate efforts to practically implement decoupling in different quantum information processors.

Conclusion.– We showed how to achieve noise-tolerant universal quantum control on the full Hilbert space of the system based on purely unitary open-loop manipulations. From the perspective of quantum information processing, this implies the potential of accomplishing noise-protected universal quantum computation without the cost of extra space resources. The method, which is best suited for slow-response non-Markovian quantum baths, complements existing approaches based on quantum error correction, where typically a memoryless error scenario is assumed. As a further step toward the goal of a truly fault-tolerant computation scheme, the present assumption of perfect control resources should be relaxed to allow *noisy* decoupling and programming operations. A separate analysis of the issue of robustness shall be presented in a future work.

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